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Construction of (1 + 1)-Dimensional Field Models with Exactly Solvable Fluctuation Equations about Classical Finite-Energy Configurations

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A systematic construction of (1 + 1)-dimensional field theoretic models with exactly solvable excitation spectrum about its classical static finite-energy configuration is presented. The approach is based on the concept of shape-invariant potentials in supersymmetric quantum mechanics. Stable and unstable field models are taken into account. In the case of stable models two new field potentials are found, which give rise to exactly solvable fluctuation equations about their topologically non-trivial classical solutions. In the case of unstable models two new families of field potentials with fractional powers and one with a logarithmic term are found. © 1997 Academic Press

1. INTRODUCTION

In the last two decades, field theoretic models with classical finite-energy solutions have attracted much attention [1]. In higher space dimension the number of such classical solutions are very limited and only a few are known so far. The best known examples are the non-linear $O(3)$ model in $(2+1)$ dimensions and the 'tHooft–Polyakov monopole in $(3+1)$ dimensions. However, in $(1+1)$ dimensions a variety of classical finite-energy solutions are known. The most prominent examples are the soliton solution of the sine-Gordon model and the kink solution of the ϕ^4-model. These topologically non-trivial solutions also play an important role in tunneling phenomena of bistable and unstable quantum systems [2, 3]. Solitons are also an essential ingredient of non-linear wave equations such as the Korteweg–de Vries and non-linear Schrödinger equation [4, 5].

These classical finite-energy solutions occur in quantum field theories with spontaneously broken symmetry. Also under certain circumstances, they may be interpreted as quantum extended-particle states. The stability of these states is analyzed by considering the (quantum-) fluctuations about the classical solution [6]. If we

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restrict ourselves to the static limit, that is, time-independent classical solutions, then the equation determining the stability of these solutions is a Schrödinger-type equation. The eigenfunctions of the corresponding fluctuation operator are then interpreted as quantum excitations of the extended particle. Clearly, one is interested in this particle spectrum.

It is the aim of this paper to construct $(1+1)$-dimensional field theory models which admit topological non-trivial classical solutions and whose stability equation is exactly solvable. This problem is, of course, not entirely new and related ideas have been studied before [7, 8]. Similar attempts are also due to Kumar [9] and Boya and Casahorran [10]. Kumar starts with the stability equation associated with the ϕ^4-kink and, using ideas of supersymmetric (SUSY) quantum mechanics, constructs a new stability equation with known spectral properties. However, the corresponding field potential could not be put into a closed form. On the other hand Boya and Casahorran start with a given form of field models with a polynomial interaction and use SUSY methods to find new kinks with exactly known particle spectrum. Fluctuation equations within the framework of SUSY quantum mechanics (both on the real line as well as on the unit circle) have also been considered by Kulshreshtha, Liang and Müller-Kirsten [11].

We do not limit ourselves to any particular form of the field potential. In fact, in the present approach we start with a given family of fluctuation equations, whose spectral properties (both the discrete as well as the continuous spectrum) are explicitly known. In this respect ideas of SUSY quantum mechanics [12, 13] will be the guiding line. Then we try to construct the corresponding field potential in closed form. This will only be possible for certain members of the family of fluctuation operators. We will consider models which posses stable and unstable classical configurations. In the later case, however, we limit ourselves to those unstable configurations, whose fluctuation operator has only one unstable mode, that is, the fluctuation operator has one negative eigenvalue. In this context only the discrete spectrum of the fluctuation operator is relevant. Hence, throughout this paper we will draw our attention on the discrete spectrum of the fluctuation operator. To simplify the program further we shall consider only static finite-energy solutions.

This paper is organized as follows. In the next section we will briefly discuss the static finite-energy solutions of a classical Lorentz-invariant field model of a scalar field in $(1+1)$ dimensions. Particular attention is given to the stability of these solutions. The corresponding stability equation is identical in form with a one-dimensional stationary Schrödinger equation.

As already mentioned, we are interested in finding field models for which the stability equation have known spectral properties. Therefore, as an ansatz we will consider a particular family of potentials [14] for which the Schrödinger equations on the real line are exactly solvable. This family, in essence, was constructed by Infeld and Hull [15] via the factorization method which goes back to Schrödinger [16]. Within the SUSY quantum mechanics this factorization has been reconsidered and is now known under the name of shape-invariant potentials [14]. In Section III we will review SUSY quantum mechanics and the concept of

shape-invariance, which provides actually four families of exactly solvable Schrödinger problems on the real line.

In Section IV these four families are then systematically investigated to obtain closed-form expressions of the corresponding field model. Besides the well-known sine-Gordon, ϕ^4 and double-quadratic model [1, 5] we find two new field potentials whose excitations about the classical finite-energy solution are exactly known.

In Section V we start with the same four families of shape-invariant potentials and construct unstable field theoretic models with exactly known particle spectrum. Here we will find two new families of field potential with (in general) fractional powers of the field. As special cases they contain the inverted double-well, a ϕ^5 and a cubic field potential. In addition we also find a model with field potential containing a logarithmic term.

II. CLASSICAL FINITE-ENERGY CONFIGURATIONS

Let us now consider $(1+1)$-dimensional field models of a scalar field $\phi(x, t)$ which exhibit classical finite-energy solutions. These models are characterized by the Lorentz-invariant Lagrangian density

$$\mathscr{L} := \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - U(\phi), \tag{2.1}$$

where U is the field potential. The corresponding classical equation of motion reads

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial U}{\partial\phi}. \tag{2.2}$$

With a solution of this equation one can associate a conserved (time-independent) energy functional

$$E[\phi] := \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + U(\phi)\right]. \tag{2.3}$$

As we are only interested in configurations with finite energy we are led to the boundary conditions $\partial\phi/\partial t \to 0$, $\partial\phi/\partial x \to 0$ and $U(\phi) \to 0$ as $x \to \pm\infty$. Consequently, such solutions approach a constant value at infinity:

$$\phi_\pm := \lim_{x \to \pm\infty} \phi(x, t) \qquad \text{with} \quad U(\phi_\pm) = 0. \tag{2.4}$$

These localized classical solutions are called *solitary waves* if they can be put into the form $\phi(x, t) = f(x - vt)$. Note that solitary waves can be obtained from static (i.e. time-independent) solutions ϕ_{st} of (2.2) via a Lorentz-boost. Because of this,

from now on we will draw our attention exclusively to those localized static solutions obeying

$$\frac{1}{2}\left(\frac{d\phi_{st}}{dx}\right)^2 = U(\phi_{st}(x)), \tag{2.5}$$

which follows from (2.2) via integration. Note that the integration constant necessarily vanishes due to the finiteness of the energy functional

The dynamical stability of the static solution can be investigated by looking at its fluctuations ψ with $\lim_{x \to \pm\infty} \psi(x) = 0$:

$$E[\phi_{st} + \psi] \approx E[\phi_{st}] + \delta E[\psi]; \tag{2.6}$$

here

$$\delta E[\psi] := \frac{1}{2} \int_{-\infty}^{+\infty} dx\, \psi(x) \left[-\frac{d^2}{dx^2} + \frac{\partial U}{\partial \phi}(\phi_{st}(x)) \right] \psi(x). \tag{2.7}$$

Hence, the stability of ϕ_{st} is controlled by the eigenvalues of the *fluctuation operator*

$$H := -\frac{d^2}{dx^2} + \frac{\partial U}{\partial \phi}(\phi_{st}(x)). \tag{2.8}$$

To be more precise, let $\{\psi_n\}$ be the complete set of eigenfunctions of H with eigenvalues $\{\mu_n\}$. Then the variation of the energy functional is given by

$$\delta E[\psi] = \tfrac{1}{2} \sum_n |a_n|^2 \mu_n, \qquad a_n := \int_{-\infty}^{+\infty} dx\, \psi_n^*(x)\, \psi(x). \tag{2.9}$$

Hence, the static solution is stable if all eigenvalues of H are non-negative, $\mu_n \geqslant 0$. If there exists at least one negative eigenvalue then ϕ_{st} will be an unstable configuration. Note that for the continuous part of the spectrum the sum in (2.9) should be replaced by an appropriate integral. Throughout this paper we will focus our attention only on the discrete part of the spectrum of the fluctuation operator (2.8).

Let us also mention that H always has a vanishing eigenvalue $\mu = 0$ due to translational invariance. The corresponding eigenstate is given by

$$\psi_{\mu=0}(x) \propto \frac{d\phi_{st}}{dx}(x). \tag{2.10}$$

As already stated earlier, the aim of this paper is to construct field models, that is, finding the field potential U, such that the eigenvalue problem for the fluctuation operator is exactly solvable. Due to the fact that H has a vanishing eigenvalue, methods developed for SUSY quantum mechanics seem to be most profitable for that purpose.

III. BASICS ON SUSY QUANTUM MECHANICS

Witten's model [17, 13] of SUSY quantum mechanics consists of a pair of standard Schrödinger–Hamiltonians $H_\pm$,

$$H_- = -\frac{d^2}{dx^2} + V_-(x) = A^\dagger A, \tag{3.1}$$

$$H_+ = -\frac{d^2}{dx^2} + V_+(x) = AA^\dagger, \tag{3.2}$$

where

$$V_\pm(x) := W^2(x) \pm W'(x), \qquad A := \frac{d}{dx} + W(x). \tag{3.3}$$

The Witten model is thus uniquely characterized by the so-called SUSY potential W, which is a real-valued function of the cartesian degree of freedom $x \in \mathbb{R}$. For unbroken SUSY one of these Hamiltonians is required to have a ground state with vanishing eigenvalue. Let us assume that this ground state, denoted by ψ_0^-, belongs to H_-, that is, $A\psi_0^- = 0$. This state can explicitly be expressed in terms of the SUSY potential,

$$\psi_0^-(x) := N \exp\left\{-\int_0^x dz\, W(z)\right\}. \tag{3.4}$$

Because of SUSY, the set of the strictly positive eigenvalues of H_- forms the complete spectrum of H_+. To be more explicit, let us assume, that the discrete spectrum of H_- is given by the set $\{\lambda_n\}_{n=0,1,\ldots,p}$, where $\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_p$, with corresponding eigenstates $\psi_0^-, \psi_1^-, \ldots, \psi_p^-$. Then the discrete spectrum of H_+ is given by $\{\lambda_1, \ldots, \lambda_p\}$ with eigenstates $\psi_1^+, \ldots, \psi_p^+$, which may be obtained from those of H_- via the SUSY transformation ($n = 1, 2, \ldots, p$)

$$\psi_n^+(x) = \frac{1}{\sqrt{\lambda_n}} A\psi_n^-(x), \qquad \psi_n^-(x) = \frac{1}{\sqrt{\lambda_n}} A^\dagger \psi_n^+(x). \tag{3.5}$$

Similar relations also hold for a possible continuous spectrum.

Let us now assume that we have a family of SUSY potentials $\{W(a_s, x)\}$, $s = 0, 1, 2, \ldots, p$, which differ only in their values of some potential parameter a_s and the corresponding ground states (3.4) are all normalizable. That is, SUSY remains unbroken for all $s = 0, 1, \ldots, p$. If now the corresponding set of partner potentials $\{V_+(a_s, x), V_-(a_s, x)\}$ obeys the shape-invariance condition [14] $V_+(a_s, x) = V_-(a_{s+1}, x) + R(a_{s+1})$, where $R(a_{s+1})$ does not depend on x, then the eigenvalue problems for the corresponding Schrödinger Hamiltonians are exactly solvable due

TABLE I

Known Shape-Invariant SUSY Potentials on the Real Line

Case	SUSY potential	Parameters
I	$W(x) = a \tanh x + b/\cosh x$	$a > 0$
II	$W(x) = a \tanh x + b/a$	$a > b \geq 0$
III	$W(x) = a - be^{-x}$	$a, b > 0$
IV	$W(x) = ax$	$a > 0$

to the SUSY transformation (3.5). For example, the eigenvalues and eigenstates for H_- with $V_- = V_-(a_0, x)$ are given by [13]

$$\lambda_n = \sum_{s=1}^{n} R(a_s), \qquad \psi_n^-(x) = \frac{A_0^\dagger}{\sqrt{\lambda_n - \lambda_0}} \cdots \frac{A_{n-1}^\dagger}{\sqrt{\lambda_n - \lambda_{n-1}}} \psi_0^-(a_n, x), \tag{3.6}$$

where

$$\psi_0^-(a_n, x) = N \exp\left\{ -\int_0^x dz\, W(a_n, x) \right\}, \qquad A_s^\dagger := -\frac{d}{dx} + W(a_s, x). \tag{3.7}$$

In Table I we list all four, so far known, SUSY potentials on the real line, which give rise to a family of shape-invariant partner potentials [18]. Table II summarizes some spectral properties of H_- for the SUSY potentials listed in Table I. Note that for case I and II only the parameter choice $b = 0$ will be relevant for the discussion below. For further details we refer to [13].

In the following section we will identify H_- with the fluctuation operator for a classical finite-energy solution. Thus the list in Table I provides us with exactly solvable models for the fluctuation operator. The aim of this paper is to obtain the corresponding field potential $U(\phi)$ in closed form. Clearly, we do expect that this will be possible only for a few parameter values a and b of the shape-invariant SUSY potentials given in Table I.

TABLE II

Discrete Eigenvalues λ_n and the Ground-State Wave Function $\psi_0^-(a, x)$ for the Hamiltonian H_- Associated with the SUSY Potentials Given in Table I

Case	Parameters	λ_n		$\psi_0^-(a, x)/N$
I + II	$a > 0, \quad b = 0$	$a^2 - (a-n)^2$,	$n = 0, 1, ..., < a$	$\cosh^{-a} x$
III	$a, b > 0$	$a^2 - (a-n)^2$,	$n = 0, 1, ..., < a$	$e^{-ax} \exp\{-be^{-x}\}$
IV	$a > 0$	$2na$,	$n = 0, 1, 2, ...$	$\exp\{-ax^2/2\}$

IV. CONSTRUCTION OF STABLE FIELD POTENTIALS

In this section we will construct stable field models with exactly solvable stability equation such that the field potential can be given in a closed form. We will start with a given shape-invariant SUSY potential taken from Table I and evaluate the corresponding SUSY ground state for H_- according to (3.4). Identifying the fluctuation operator (2.8) with H_-, $H=H_-$, this SUSY ground state is also an eigenstate of H with vanishing eigenvalue $\lambda_0=\mu_0=0$. In a second step we then determine the explicit form of the static finite-energy solution via integration of (2.10),

$$\phi_{st}(x)=\alpha\int_0^x dz\,\psi_0(z)+\beta, \tag{4.1}$$

where α and β are some real-valued constants. This step provides us with a first criterion for the parameters a and b of the SUSY potential (cf. Table I), because not for all possible values an explicit integration may be performed. From this solution one can finally find the field potential via the relation (2.5),

$$U(\phi_{st})=\frac{1}{2}\left(\frac{d\phi_{st}}{dx}\right)^2. \tag{4.2}$$

Clearly, for this one needs to find $x=x(\phi_{st})$. Hence, a second criterion appears. Only for those cases one is able to find $U(\phi_{st})$, which still does not provide the full field potential. Note that $\phi_{st}\in(\phi_-,\phi_+)$. However, once the function $U(\phi_{st})$ is know explicitly as a function of ϕ_{st} the full potential may simply be found via continuation. This, of course, may be done by some educated guess.

The above mentioned criteria are very restrictive and will lead to only a few parameter sets $\{a,b\}$ for which the program can actually be carried out. Let us now investigate the four SUSY potentials given in Table I.

A. *Cases I and II*

We will consider the cases I and II simultaneously as they give rise to the same field models. The zero modes are easily calculable for any value of the parameters a and b:

$$\begin{aligned}\psi_0(x)&=N\frac{\exp\{-b\arcsin(\tanh x)\}}{\cosh^a x} && \text{for Case I,}\\ \psi_0(x)&=N\frac{\exp\{-bx/a\}}{\cosh^a x} && \text{for Case II.}\end{aligned} \tag{4.3}$$

However, the explicit integration (4.1) may in both cases only be performed for integer a and vanishing b. Because of the latter condition these two cases become identical. The resulting static solutions are for example given in Eq. (2.40) of the paper by Boya and Casahorran [10]. In essence, these solutions are polynomials in

$\tanh x$ and $1/\cosh x$ for even and odd a, respectively. The degree of these polynomials is $a-1$. Obviously, only for the lowest values of a one can hope to find $x=x(\phi_{st})$ explicitly. Below we will present our results for $a=1$, 2, 3, 4 and a limiting case. The corresponding discrete eigenvalues of the fluctuation operator are given in Table II because $\mu_n=\lambda_n$ and the eigenmodes may be derived from the state $\psi_0^-(a-n, x)$ (see Table II) via (3.6) as $\psi_n=\psi_n^-$.

(i) *The sine-Gordon model, $a=1$:* In this case the static solution is given by the well-known soliton

$$\phi_{st}(\mathrm{x})=2\arcsin(\tanh x), \qquad \phi_{\pm}=\pm\pi. \tag{4.4}$$

if we set $\alpha N=2$ and $\beta=0$. The corresponding field potential reads $U(\phi_{st})=2/\cosh^2 x=2(1-\sin^2(\phi_{st}/2))$ and its analytical continuation beyond $\phi_{\pm}$ leads to the well-known sine-Gordon model

$$U(\phi)=1+\cos(\phi). \tag{4.5}$$

(ii) *The ϕ^4-model, $a=2$:* Here the static solution is also well known and reads for $\alpha N=1$ and $\beta=0$

$$\phi_{st}(x)=\tanh x, \qquad \phi_{\pm}=\pm 1. \tag{4.6}$$

For the field potential we find $U(\phi_{st})=1/2\cosh^4 x=\frac{1}{2}(1-\phi_{st}^2)^2$, which leads upon analytic continuation to the ϕ^4-model

$$U(\phi)=\tfrac{1}{2}(1-\phi^2)^2. \tag{4.7}$$

(iii) *The unresolvable case $a=3$:* For the parameter value $a=3$ the static solution explicitly reads ($\alpha N=2$, $\beta=0$)

$$\phi_{st}(\mathrm{x})=\frac{\tanh x}{\cosh x}+\arcsin(\tanh x), \qquad \phi_{\pm}=\pm\frac{\pi}{2}. \tag{4.8}$$

Here it is impossible to solve this equation for $x=x(\phi_{st})$ and therefore, the field potential can only be given implicitly via (4.8)

$$U(\phi_{st})=\frac{2}{\cosh^6 x}=U(-\phi_{st}). \tag{4.9}$$

Only for small and very large values of x one can determine the analytic behavior of the field potential near its local maximum and its minima,

$$\begin{aligned} U(\phi)&\approx 2-\frac{3}{2}\phi^2 \qquad \text{for} \quad |\phi|\ll 1,\\ U(\phi)&\approx 2\left(\phi\mp\frac{\pi}{2}\right)^6 \qquad \text{for} \quad \phi\approx\pm\frac{\pi}{2}. \end{aligned} \tag{4.10}$$

(iv) *A new model for* $a=4$: Whereas for $a=3$ it was impossible to find an explicit analytical expression for the field potential, it is still possible in the case $a=4$. Here the static solution reads for $\alpha N=1$ and $\beta=0$

$$\phi_{st}(x)=\tanh x-\tfrac{1}{3}\tanh^3 x, \qquad \phi_{\pm}=\pm\tfrac{2}{3}. \tag{4.11}$$

This cubic equation can be solved for $\tanh x$, which will be sufficient to calculate the field potential explicitly. The physical allowed solution is

$$\tanh x=-2\cos\left(\frac{1}{3}\arccos\left(\frac{3}{2}\phi_{st}\right)+\frac{4\pi}{3}\right), \tag{4.12}$$

where $\arccos z\in[0,\pi]$. The field potential reads

$$U(\phi_{st})=\frac{1}{2}(1-\tanh^2 x)^4=\frac{1}{2}\left[1-4\cos^2\left(\frac{1}{3}\arccos\left(\frac{3}{2}\phi_{st}\right)+\frac{4\pi}{3}\right)\right]^4. \tag{4.13}$$

This potential may, for example, periodically be continued beyond $\phi_{\pm}$ by setting $U(\phi+4/3)=U(\phi)$ and

$$U(\phi)=\frac{1}{2}\left[1+2\cos\left(\frac{2}{3}\arccos\left(\frac{3}{2}\phi\right)+\frac{8\pi}{3}\right)\right]^4 \tag{4.14}$$

for $-2/3\leqslant\phi\leqslant 2/3$.

(v) *The double-quadratic model as limiting case*: Finally, let us consider the SUSY potential $W(x)=a\,\mathrm{sgn}(x)$, $a>0$, which may be considered as a limiting case of the previous ones in the sense that $W(x)=\lim_{\gamma\to\infty} a\tanh(\gamma x)$. Note that $V_{\pm}(x)=a^2\pm 2a\delta(x)$ for which the eigenvalue problem is easily solvable [19]. It has only one bound state and the corresponding ground-state wave function reads $\psi_0(x)=Ne^{-a|x|}$ from which the finite-energy solution immediately follows ($\alpha N=1$, $\beta=0$):

$$\phi_{st}(x)=\frac{\mathrm{sgn}\,x}{a}(1-e^{-a|x|}), \qquad \phi_{\pm}=\pm\frac{1}{a}. \tag{4.15}$$

The corresponding field model is that for the well-known double-quadratic potential (see, for example, Ref. [5, p. 412])

$$U(\phi)=\tfrac{1}{2}(1-a|\phi|)^2. \tag{4.16}$$

In Figs. 1 and 2 for all of the above solvable cases the curves for the static finite-energy solutions and the field potentials are given, respectively.

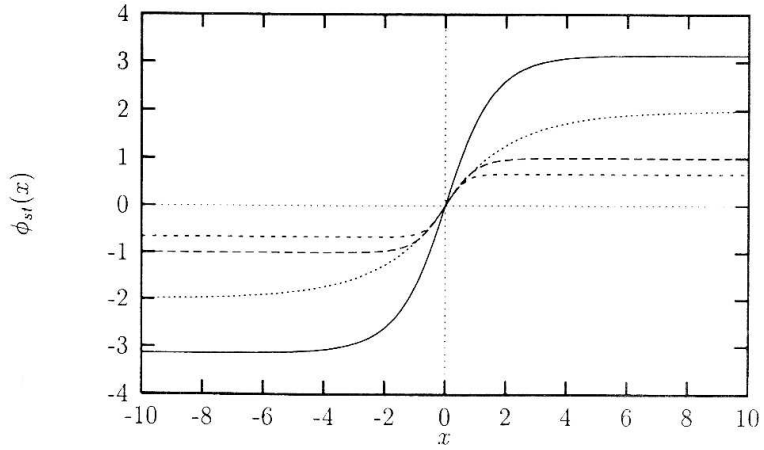

FIG. 1. The static solutions for the stable field models corresponding to cases I and II. Shown are the classical static finite-energy solutions for the sine-Gordon model $a=1$ (—), the ϕ^4-model $a=2$ (– – –), the new potential (4.14) obtained for $a=4$ (- - -) and for the double-quadratic case (4.15) with $a=0.5$ (· · ·).

B. *Cases III and IV*

Now we will investigate the shape-invariant SUSY potential belonging to case III and IV of Table I. For case III the SUSY ground state is given in Table II. In this case an analytic expression for the static solution can only be found if we set $a=1$:

$$\phi_{st}(x)=\frac{1}{b}\exp\{-be^{-x}\}, \qquad \phi_{-}=0, \quad \phi_{+}=\frac{1}{b}; \tag{4.17}$$

here we have chosen $\alpha=b/N$ and $\beta=\exp\{-b\}$ for convenience. Note that we have no restriction on the potential parameter b. The corresponding field potential as a function of the static solution reads

$$U(\phi_{st})=\tfrac{1}{2}(e^{-x}\exp\{-be^{-x}\})^2=\tfrac{1}{2}\phi_{st}^2\ln^2(b\phi_{st}). \tag{4.18}$$

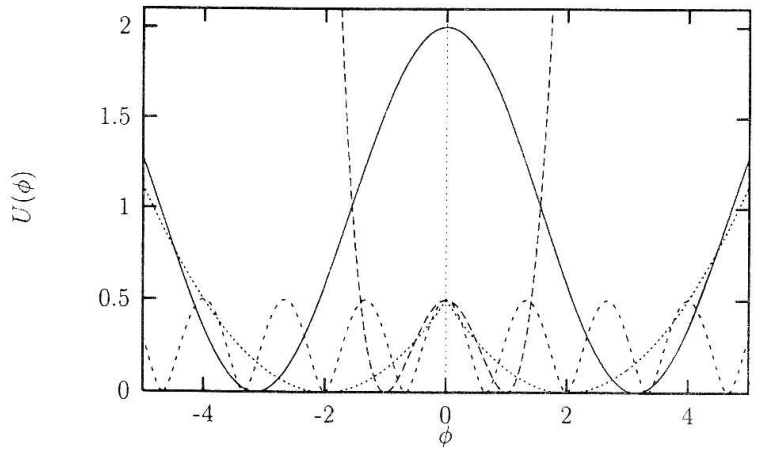

FIG. 2. The field potentials for the same cases as in Fig. 1.

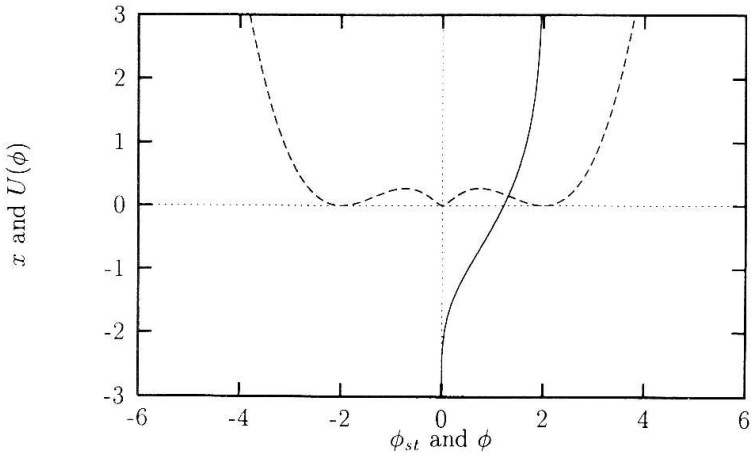

FIG. 3. The static solution (solid line) and the corresponding stable field potential (dashed line) for case III with $b = 0.5$.

This potential may easily be continued beyond the field values taken by ϕ_{st}. Here we simply choose the absolute value in the argument of the logarithm. This leads to a new family of field models

$$U(\phi) = \tfrac{1}{2}\phi^2 \ln^2(b\,|\phi|), \qquad b > 0. \tag{4.19}$$

In Fig. 3 we present the shape of the static classical finite-energy solution together with the field potential for $b = 0.5$. The discrete eigenvalues of the fluctuation operator are given in Table II. Actually, because of $a = 1$, there exists only one, namely the ground-state eigenvalue $\mu_0 = \lambda_0 = 0$.

Finally, for the last case IV one cannot find a closed form expression for the field potential because here the static solution is given by an error function ($\alpha = 1/N$, $\beta = 0$),

$$\phi_{st}(x) = \sqrt{\frac{\pi}{2a}}\ \mathrm{Erf}(x\sqrt{a/2}), \tag{4.20}$$

and does not allow to solve for $x = x(\phi_{st})$.

V. CONSTRUCTION OF UNSTABLE FIELD POTENTIALS

The aim of this section is to obtain unstable field models whose stability equation is also exactly solvable. Unstable means that there exists at least one negative eigenvalue of the fluctuation operator (2.8). Here we will construct only those models which have exactly one unstable mode. In other words, the fluctuation operator has one negative eigenvalue $\mu_0 < 0$. Its first excited state belongs to the translational mode having the vanishing eigenvalue $\mu_1 = 0$.

Our construction principle is also based on SUSY quantum mechanics. Here, however, we have to identify the fluctuation operator H with the shifted SUSY Hamiltonian, $H = H_- + \mu_0 = H_- - \lambda_1$. Consequently their eigenvalues are related by $\mu_n = \lambda_n + \mu_0$ and the corresponding eigenstates are identical $\psi_n = \psi_n^-$. Now the first excited state $\psi_1 = \psi_1^-$ of H will serve to find the static solution in analogy to (2.10). This state is obtained from the SUSY ground state of the partner Hamiltonian via the SUSY transformation (3.6):

$$\psi_1(x) = A_0^\dagger \psi_0^-(a_1, x). \tag{5.1}$$

Hence our starting point is a pair of shape-invariant SUSY potentials $\{W(a_0, x), W(a_1, x)\}$. In terms of these SUSY potentials the above relation explicitly reads

$$\psi_1(x) = N[W(a_1, x) + W(a_0, x)] \exp\left\{-\int_0^x dz\, W(a_1, z)\right\}. \tag{5.2}$$

Then the static solution is given by

$$\phi_{st}(x) = \alpha \int_0^x dz\, \psi_1(z) + \beta \tag{5.3}$$

and from this the field potential is obtained via (2.5).

As in the previous section, we have to face the same two conditions for the parameters $a = a_0$ and b. These are the explicit integrability of (5.3) and the solvability of $x = x(\phi_{st})$. Here, however, it turns out that these conditions are not as restrictive as in the case of stable field models. In particular the second condition will not lead to any further restrictions. Again we will investigate the four SUSY potentials listed in Table I.

A. *Cases I and II*

As before, we will consider cases I and II simultaneously as they finally lead to the same field models. The pair of SUSY potentials for case I reads ($a_0 = a > 1$, $a_1 = a - 1$)

$$\begin{aligned} W(a_0, x) &= a \tanh x + \frac{b}{\cosh x}, \\ W(a_1, x) &= (a-1) \tanh x + \frac{b}{\cosh x}. \end{aligned} \tag{5.4}$$

The corresponding first excited state of H is then given by

$$\psi_1(x) = N\left((2a-1)\tanh x + \frac{2b}{\cosh x}\right) \frac{\exp\{-b \arcsin(\tanh x)\}}{\cosh^{a-1} x}. \tag{5.5}$$

Similarly, in case II we start with ($a_0 = a > 1$, $a_1 = a - 1$, $b \geqslant 0$)

$$W(a_0, x) = a \tanh x + \frac{b}{a},$$
$$W(a_1, x) = (a-1) \tanh x + \frac{b}{a-1}, \tag{5.6}$$

and find

$$\psi_1(x) = N\left((2a-1)\tanh x + b\left(\frac{1}{a} + \frac{1}{a-1}\right)\right)\frac{\exp\{-bx/(a-1))\}}{\cosh^{a-1} x}. \tag{5.7}$$

In both cases it turns out that (5.3) can only be integrated explicitly for $b = 0$. Therefore, cases I and II again become identical. The resulting static solution reads

$$\phi_{st}(x) = \frac{1}{\cosh^{a-1} x}, \tag{5.8}$$

where we have set $\alpha N = (1-a)/(2a-1)$ and $\beta = -N(2a-1)/(a-1)$. The simple relation (5.8) allows to find $x = x(\phi_{st})$ for all values of $a > 1$ and with

$$\frac{d\phi_{st}}{dx} = (1-a)\frac{\sinh x}{\cosh^a x} = (1-a)\,\phi_{st}^{a/(a-1)} \sinh x$$

we obtain the field potential

$$U(\phi_{st}) = \frac{1}{2}\left(\frac{d\phi_{st}}{dx}\right)^2 = \frac{(a-1)^2}{2}\phi_{st}^2(1 - \phi_{st}^{(a+1)/(a-1)}), \qquad 0 < \phi_{st} \leqslant 1. \tag{5.9}$$

There are at least two ways to continue the fractional power in the above expression beyond the values taken by ϕ_{st},

$$U_1(\phi) = \frac{(a-1)^2}{2}\phi^2(1 - |\phi|^{(a+1)/(a-1)}), \qquad a > 1, \tag{5.10}$$

$$U_2(\phi) = \frac{(a-1)^2}{2}\phi^2(1 - \operatorname{sgn}\phi\ |\phi|^{(a+1)/(a-1)}), \qquad a > 1, \tag{5.11}$$

leading to two new families of unstable field potentials with, in general, fractional powers of the field. Note that $a \in \mathbb{R}$ with $a > 1$. For the particular values $a = 2$, 3 and $a \to \infty$, however, these reduce to polynomial potentials. For example, U_2 for $a = 2$ reduces to an unstable ϕ^5-theory, $U_2(\phi) = \frac{1}{2}\phi^2(1 - \phi^3)$. For the case $a = 3$ the other choice gives rise to the inverted double-well potential $U_1(\phi) = 2\phi^2(1 - \phi^2)$. In

the limit of large a, $a \to \infty$, we also recover the ϕ^3-model, $\lim_{a\to\infty} U_2(\phi)/a^2 = \frac{1}{2}\phi^2(1-\phi)$. Finally, let us give the discrete spectrum of the fluctuation operator:

$$\mu_n = (a-1)^2 - (a-n)^2, \qquad n = 0, 1, 2, \ldots < a. \tag{5.12}$$

In Fig. 4 we present the shape of the static solution (5.8) for $a = 2$, 3 and 4. Figs. 5 and 6 show the corresponding unstable field potentials U_1 and U_2, respectively.

B. *Cases III and IV*

For case III the starting pair of shape-invariant SUSY potentials reads ($a_0 = a > 1$, $a_1 = a - 1$, $b > 0$)

$$W(a_0, x) = a - be^{-x}, \qquad W(a_1, x) = (a-1) - be^{-x}, \tag{5.13}$$

and the first excited state of H is given by

$$\psi_1(x) = N((2a-1) - 2be^{-x})\, e^{-(a-1)x} \exp\{-be^{-x}\}. \tag{5.14}$$

Unfortunately, for none of the allowed potential parameters a and b one can find the corresponding static solution (5.3) in closed form.

Therefore, let us pass to the last case IV. Here the starting pair of SUSY potentials is ($a_0 = a_1 = a > 0$)

$$W(a_0, x) = W(a_1, x) = ax, \tag{5.15}$$

which leads to

$$\psi_1(x) = 2Naxe^{-ax^2/2}. \tag{5.16}$$

Hence the static solution in this case reads ($\alpha = 1/2N$, $\beta = 1$)

$$\phi_{st}(x) = \exp\{-ax^2/2\} \tag{5.17}$$

and $x^2 = -(2/a)\ln \phi_{st}$. Therefore, the field potential can explicitly be computed, $U(\phi_{st}) = -a\phi_{st}^2 \ln \phi_{st}$ and is easily continued by taking the absolute value for the argument of the logarithm:

$$U(\phi) = -a\phi^2 \ln|\phi|, \qquad a > 0. \tag{5.18}$$

The eigenvalues of the fluctuation operator are given by

$$\mu_n = 2a(n-1), \qquad n = 0, 1, 2, \ldots. \tag{5.19}$$

Figure 7 gives the shape of the static solution (5.17) and the corresponding field potential (5.18) for $a = 5$.

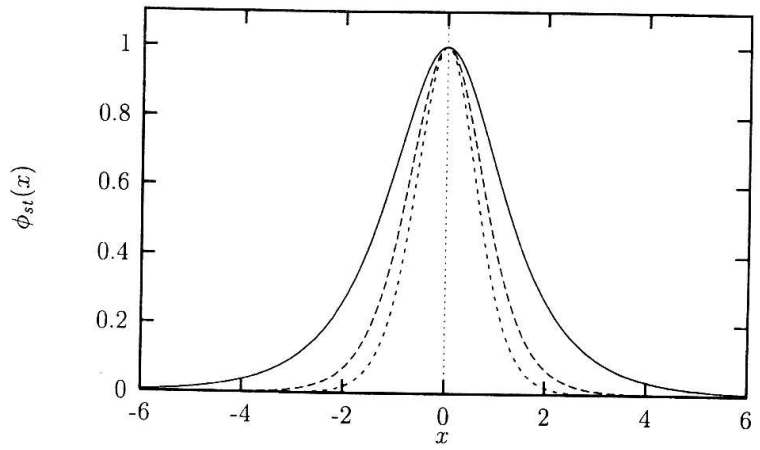

FIG. 4. The static finite-energy solution for the unstable field potentials corresponding to case I and II for $a=2$ (—), $a=3$ (– – –) and $a=4$ (- - -).

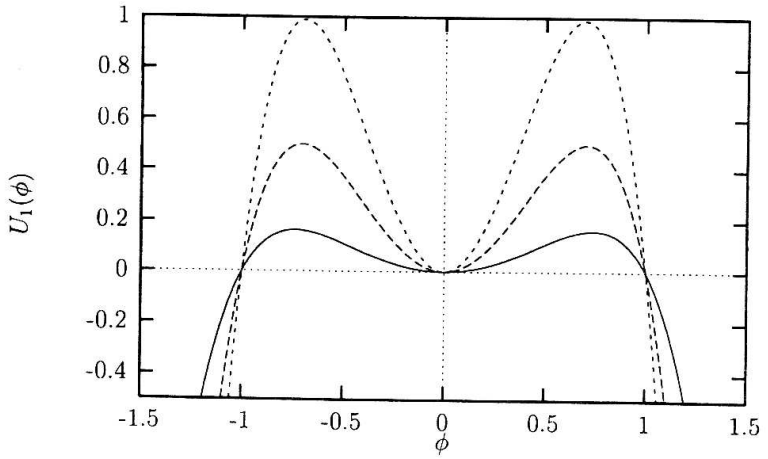

FIG. 5. The unstable field potentials $U_1(\phi)$ corresponding to cases I and II. Parameters are the same as in Fig. 4.

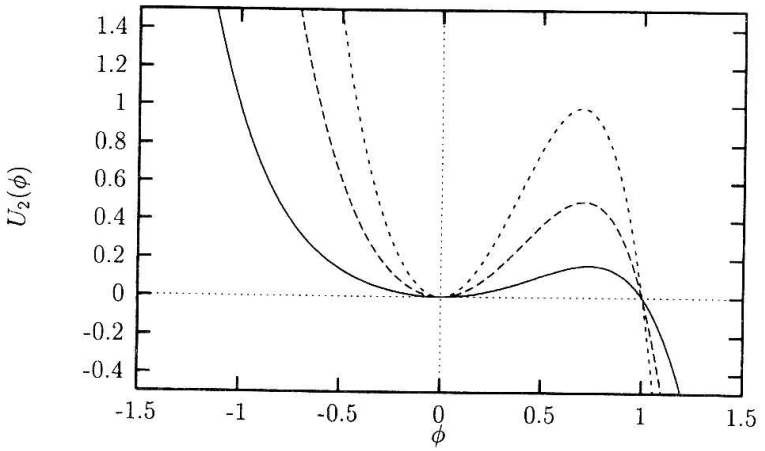

FIG. 6. The unstable field potentials $U_2(\phi)$ corresponding to cases I and II. Parameters are the same as in Fig. 4.

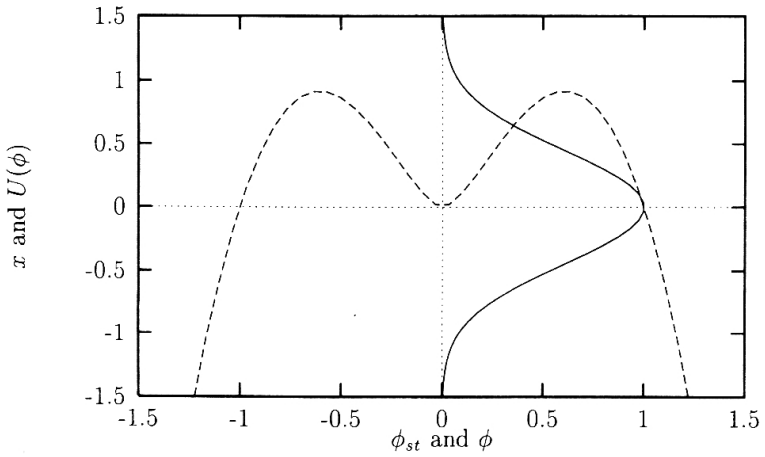

FIG. 7. The static solution (solid line) and the corresponding unstable field potential (dashed line) for case IV with $a = 5$.

VI. CONCLUDING REMARKS

In this paper we have used ideas of SUSY quantum mechanics to construct stable and unstable field theory models in $(1 + 1)$ dimensions which admit topological non-trivial classical finite-energy configurations (solitary waves). The basic idea was to start with families of one-dimensional quantum mechanical potentials such that the corresponding Schrödinger-like eigenvalue problem is exactly solvable. These families are provided by the recent investigation of shape-invariant systems within the framework of SUSY quantum mechanics. Identifying the Schrödinger Hamiltonian with the fluctuation operator for the static classical finite-energy solution of a $(1 + 1)$-dimensional field theory we tried to construct the corresponding field potential in a closed form. In doing so, we had to face restrictions on the parameters of the starting SUSY potentials listed in Table I.

For the case of stable field models, as discussed in Section IV, these restrictions led us to only five explicit field potentials. Besides the known sine-Gordon, ϕ^4 and

TABLE III

Stable Field Potentials Found in Section IV

Case	Parameters	Field potential	Model
I + II	$a = 1, \quad b = 0$	$U(\phi) = 1 + \cos\phi$	sine-Gordon
	$a = 2, \quad b = 0$	$U(\phi) = \frac{1}{2}(1 - \phi^2)^2$	ϕ^4-theory
	$a = 4, \quad b = 0$	$U(\phi) = (4.14)$	New
	Limiting case	$U(\phi) = \frac{1}{2}(1 - a\lvert\phi\rvert)^2$	Double-quadratic
III	$a = 1, \quad b > 0$	$U(\phi) = \frac{1}{2}\phi^2 \ln^2(b\lvert\phi\rvert)$	New
IV	none	none	none

TABLE IV

Unstable Field Potentials Found in Section V

Case	Parameters	Field potential	Model
I + II	$a>1, \quad b=0$	$U_1(\phi)=\frac{(a-1)^2}{2}\phi^2(1-\lvert\phi\rvert^{(a+1)/(a-1)})$	New
	$a>1, \quad b=0$	$U_2(\phi)=\frac{(a-1)^2}{2}\phi^2(1-\operatorname{sgn}\phi\ \lvert\phi\rvert^{(a+1)/(a-1)})$	New
III	None	None	None
IV	$a>0$	$U(\phi)=-a\phi^2\ln\lvert\phi\rvert$	New

double-quadratic model we found two new field potentials. These results are summarized in Table III.

For the unstable field models a much larger range of parameters is allowed and it leads to explicit field potentials. We have obtained two new families of field potentials, U_1 and U_2, associated with the SUSY potential $W(x)=a\tanh x$, $a>1$. They contain as special cases the inverted double-well potential, a ϕ^5- and a ϕ^3-model. In general, however, these field potentials have fractional powers of the fields. For the harmonic-oscillator SUSY potential $W(x)=ax$, $a>0$, a new field theory with logarithmic interaction has been found. The results of our investigation in Section V for unstable field theories are summarized in Table IV. Note that in the case of unstable field models we have limited ourselves to only those with one unstable mode. Clearly, our construction method based on SUSY quantum mechanics can also be applied to systems with two or more unstable modes of the fluctuation operator.

Finally, we would like to mention that field theories with non-polynomial interaction have been studied before [20] and it would be of interest to study the new models found here from the point of view of renormalization, triviality, etc. [21]. The present approach may also be useful in finding new solvable non-linear wave equations [22].

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